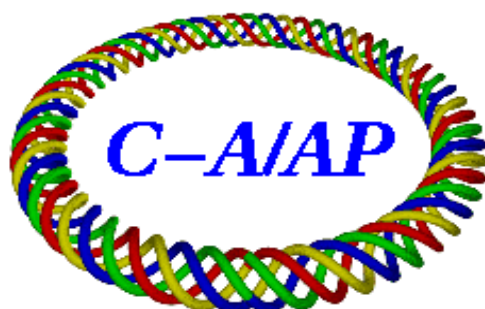# Investigation of Helical Pitch on $B_\perp$

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Waldo MacKay, 5 July 2002

The magnetic field at a point $\vec{r}$ generated by a current loop in the absence of magnetic materials may be obtained from the integral:

$$\vec{B}(\vec{r}) = -\frac{\mu_0}{4\pi} \oint \frac{(\vec{r} - \vec{r}') \times d\vec{r}'}{|\vec{r} - \vec{r}'|^3}.$$

If we consider a current I flowing down a thin wire placed on a helical path, we may write the current density in cylindrical coordinates as

$$\vec{j}(\vec{r}') = \frac{I}{a}\, \delta(r' - a)\, \delta(\theta' - kz') \left[\sin(ka)\hat{\theta} + \cos(ka)\hat{z} \right],$$

where the helix is aligned along the z-axis with radius a and and pitch $k = 2\pi/\lambda$. The helix may be parameterized in terms of z as

$$\vec{r}' = (\, a, \quad kz, \quad z\,),$$

with the path differential

$$d\vec{r}' = (ka\hat{\theta} + \hat{z})\, dz.$$

The field components at the origin are then given by

$$\vec{B}(\vec{0}) == \frac{\mu_0}{4\pi} \oint \frac{\vec{r}' \times d\vec{r}'}{|\vec{r}'|^3}.$$

For evaluating the cross product it is more obvious if we write things in terms of Cartesian coordinates:

$$\vec{r}' = (\, a\cos(\theta_0 + kz), \quad a\sin(\theta_0 + kz), \quad z\,)$$
$$d\vec{r}' = (\, -ka\sin(\theta_0 + kz), \quad ka\cos(\theta_0 + kz), \quad 1\,)\, dz.$$

Evaluating the cross product in the numerator of the integrand gives

$$\hat{r}' \times d\hat{r}' = \begin{pmatrix} [\sin(\theta_0 + kz) - kz\cos(\theta_0 + kz)] \\ -[\cos(\theta_0 + kz) + kz\sin(\theta_0 + kz)] \\ ka \end{pmatrix} a\, dz,$$

where θ_0 gives the angular orientation of the wire at $z = 0$. The denominator becomes

$$|\vec{r}'|^3 = \left[a^2 + z^2 \right]^{\frac{3}{2}}.$$

For a single infinitely long helical wire

$$B_x(0) = \frac{\mu_0 I a}{4\pi} \int_{-\infty}^{\infty} \frac{\sin(\theta_0 + kz) - kz\cos(\theta_0 + kz)}{[a^2 + z^2]^{\frac{3}{2}}}\, dz$$

$$B_y(0) = -\frac{\mu_0 I a}{4\pi} \int_{-\infty}^{\infty} \frac{\cos(\theta_0 + kz) + kz\sin(\theta_0 + kz)}{[a^2 + z^2]^{\frac{3}{2}}}\, dz.$$

In a typical magnet we have a second return current with an identical helix opposite the first current, so we may consider two helices: one with current $+I$ and $\theta_0 = 0$ and the second with current $-I$ and $\theta_0 = \pi$. Adding the fields together gives

$$B_x(0) = \frac{\mu_0 I a}{2\pi} \int_{-\infty}^{\infty} \frac{\sin(kz) - kz \cos(kz)}{[a^2 + z^2]^{\frac{3}{2}}} \, dz$$

$$B_y(0) = -\frac{\mu_0 I a}{2\pi} \int_{-\infty}^{\infty} \frac{\cos(kz) + kz \sin(kz)}{[a^2 + z^2]^{\frac{3}{2}}} \, dz.$$

With this orientation of wires, it is clear that $B_x(0) = 0$ since the integrand is an odd function of z. So we get

$$B_\perp(0) = -\frac{\mu_0 I}{\pi a} \int_{-\infty}^{\infty} \frac{a^2}{2} \frac{\cos(kz) + kz \sin(kz)}{[a^2 + z^2]^{\frac{3}{2}}} \, dz.$$

For straight wires ($k = 0$) the integrand is 1, and we have the expected value

$$B_\perp = -\frac{\mu_0 I}{\pi a}.$$

For $k \neq 0$ we may substitute $\xi = kz$ and rewrite $B_\perp(0)$ as

$$B_\perp(0) = -\frac{\mu_0 I}{\pi a} \int_{-\infty}^{\infty} \frac{(ka)^2}{2} \frac{\cos(\xi) + \xi \sin(\xi)}{[(ka)^2 + \xi^2]^{\frac{3}{2}}} \, d\xi.$$

If we define an enhancement factor by the integral

$$E(\zeta) = \int_{-\infty}^{\infty} \frac{(\zeta)^2}{2} \frac{\cos(\xi) + \xi \sin(\xi)}{[\zeta^2 + \xi^2]^{\frac{3}{2}}} \, d\xi$$

then

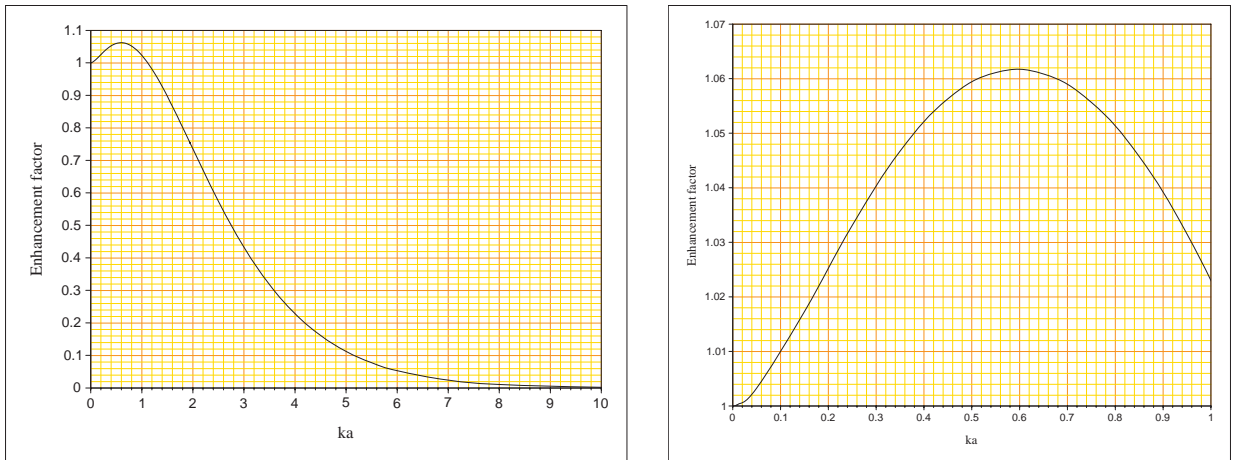$$B_\perp(0) = -\frac{\mu_0 I}{\pi a} E(ka).$$

Figure 1. Plots of the enhancement factor $E(ka)$ as calculated by MapleV.

The Enhancement factor peaks providing just over 6% more field than a straight dipole for a value of $ka \simeq 0.6$. For values of ka below this peak, the field is increased by twisting the coils; but for larger values of ka, the enhancement factor decreases asymptotically to zero. This should be expected since for extremely high pitches, the two wires look like a pair of superimposed identical solenoids with opposite currents.

Table 1. Parameters for various helical dipoles

	λ	k	a	ka
RHIC helix	2.41 m	2.607 m^{-1}	0.05—0.08 m	0.13—0.21
AGS middle	1.72 m	3.64 m^{-1}	0.1—0.13 m	0.36—0.47
AGS end	0.86 m	7.29 m^{-1}	0.1—0.13 m	0.73—0.95

Note: I have only guessed at the radial extent of the coils for the new AGS snake design.

Table 1. gives my estimate of ka for the RHIC helical dipoles and the proposed AGS snake. It is interesting that the middle and end pitches of the AGS snake bracket the peak with an average enhancement of about $\sim 4.5\%$ and $\sim 5.2\%$ respectively. It is quite possible that when a transition of pitch is made, there might be a local enhancement of the field which is a little larger than the maximum shown in Fig. 1.